Some Interesting Properties of a Novel Subclass of Multivalent Function with Positive Coefficients

Aqeel Ketab AL-khafaji  
Faculty of Education for Pure Sciences  
Ibn Al-Haytham  
University of Baghdad  
Baghdad - Iraq  
aqeelketab@gmail.com

Waggas Galib Atshan  
Faculty Of Computer Science & Information Technology  
University of Al-Qadisiyah  
Diwaniyah - Iraq  
waggas.galib@au.edu.iq  
waggashnd@gmail.com

Salwa Salman Abed  
Faculty of Education for Pure Sciences  
Ibn Al-Haytham  
University of Baghdad  
Baghdad – Iraq  
salwaalbundi@yahoo.com

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Abstract— In this paper, we introduce a new class of multivalent functions defined by \( A(p,\gamma,\omega) \) where \( A(p) \) is a subclass of analytic and multivalent functions \( W(p) \) in the open unit disc \( U = \{z:|z| < 1\} \). Moreover, we consider and prove theorems explain some of the geometric properties for such new class was \( A(p,\gamma,\omega) \), such as, coefficient estimates, growth and distortion, extreme points, radii of starlikeness, convexity and close-to-convexity as well as the convolution properties for the class \( A(p,\gamma,\omega) \).

Keywords— Multivalent function; coefficient estimates; Hadamard product; growth theorem.

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1. Introduction

Let \( W(p) \) be denote the class of functions of the form:
\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, (z \in U, p \in \mathbb{N} = \{1,2,3,...\})
\]  
(1)

which are analytic and multivalent in the open unit disc \( U = \{z:|z| < 1\} \).

Let \( A(p) \) denotes a subclass of \( W(p) \) of functions of the form:
\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (a_k \geq 0, z \in U, \ p \in \mathbb{N}_0).
\]  
(2)

The convolution [6] (Hadamard) product of two power series for the function \( f(z) \) given by (1) and \( g(z) \) given by
\[
g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \quad (z \in U, p \in \mathbb{N} = \{1,2,3,...\})
\]
Can be defined by:

\[(f \ast g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k. \, (z \in U, \, p \in \mathbb{N}) \quad (3)\]

A function \(f(z) \in A(p)\) is said to be multivalent starlike of order \(\delta\), multivalent convex of order \(\delta\) and multivalent closed to convex of order \(\delta\), \((0 \leq \delta < p, \, z \in U)\) [3], respectively if \(\text{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > \delta, \text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta\) and \(\text{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \delta\).

In the next definition, we give the condition for the function \(f\) belongs in the class \(A(p, \gamma, \omega)\).

**Definition:** A function \(f \in A(p)\) belongs to the class \(A(p, \gamma, \omega)\), if it’s satisfies the following condition:

\[\frac{z[w(z)]''}{f''(z)} - \frac{pz}{\omega z[w(z)]''} < 1, \quad \left(\frac{1}{2} \leq \omega < 1, 0 < \gamma \leq \frac{1}{2}\right) \quad (4)\]

where \(w(z) = zf'(z)\).

Such type of study was carried out by various authors for another classes, like, Khairnar and More [5], AL-khafaji et al. [1], Aouf and Mostafa [2], Raina and Srivastava [7] and Dziok and Srivastava [4].

In this paper we introduce a new class \(A(p, \gamma, \omega)\), of multivalent functions in the open unit disc. Coefficient estimates, growth and distortion theorems, radii of close-to-convexity, starlikeness and convexity, extreme points and the Hadamard product for functions in the class \(A(p, \gamma, \omega)\) are obtained.

### 2. Geometric properties for \(A(p, \gamma, \omega)\).

In this section, we introduce theorems with their proofs to discuss some of the geometric properties for such class \(A(p, \gamma, \omega)\).

#### 2.1. Coefficient estimates.

A sufficient and necessary condition to the function \(f(z)\) to be in the class \(A(p, \gamma, \omega)\) will discuss in the following theorem.

**Theorem 2.1.1.** A function \(f\) in (2) belongs to the class \(A(p, \gamma, \omega)\) if and only if

\[ \sum_{k=p+1}^{\infty} k(k-1)[k-p-\omega(k+1)+\gamma]a_k \leq p(p-1)[\omega(p+1)-\gamma]. \quad (5)\]

where \(\left(\frac{p}{2} \leq 1, \frac{1}{2} \leq \omega < 1, 0 < \gamma \leq \frac{1}{2}\right)\).

The result is sharp for the function

\[f(z) = z^p + \frac{p(p-1)[\omega(p+1)-\gamma]}{k(k-1)[k-p-\omega(k+1)+\gamma]}z^k. \quad (6)\]

**Proof:** Suppose that \(f \in A(p, \gamma, \omega)\), then by (4), we have:

\[\frac{z[pz^p + \sum_{k=p+1}^{\infty} k(k-1)\gamma(k+1)]''}{\omega [pz^p + \sum_{k=p+1}^{\infty} k(k-1)a_k z^k]''} - \frac{pz}{\omega z[pz^p + \sum_{k=p+1}^{\infty} k(k-1)a_k z^k]} < 1\]

\[= \frac{\sum_{k=p+1}^{\infty} k(k-1)k(k-2)a_k z^k}{[p(p-1)[\omega(p+1)+\gamma]z^{p-1} + \sum_{k=p+1}^{\infty} k(k-1)a_k z^k]} \leq 1.\]

Since \(\text{Re}(z) \leq |z|\) for all \(z\), we have

\[\text{Re}\left\{\frac{\sum_{k=p+1}^{\infty} k(k-1)(k-p)a_k z^{k-1}}{p(p-1)[\omega(p+1)-\gamma]z^{p-1} + \sum_{k=p+1}^{\infty} k(k-1)[\omega(k+1)+\gamma]a_k z^{k-1}}\right\} \leq 1.\]

Choosing the value of \(z\) on the real axis and letting \(z \to 1^{-}\) through values, we get:

\[\sum_{k=p+1}^{\infty} k(k-1)(k-p)a_k \leq p(p-1)[\omega(p+1)-\gamma] + \sum_{k=p+1}^{\infty} k(k-1)[\omega(k+1)+\gamma]a_k.\]

Hence

\[\sum_{k=p+1}^{\infty} k(k-1)[k-p-\omega(k+1)+\gamma]a_k \leq p(p-1)[\omega(p+1)-\gamma].\]

Conversely, assume that (5) holds \(|z| = r, r < 1\), then
Since (5) holds. So we have:

\[
\begin{align*}
\sum_{k=p+1}^{\infty} k(k-1)(k-p-\omega(k+1)+\gamma)a_k \\
- p(p-1)(\omega(p+1)-\gamma)z^{p-1} \\
+ \sum_{k=p+1}^{\infty} k(k-1)(k-p-\omega(k+1)+\gamma)z^{k-1} \\
\leq \sum_{k=p+1}^{\infty} k(k-1)(k-p)a_kz^{k-1} \\
- p(p-1)(\omega(p+1)-\gamma)|z|^{p-1} \\
- \sum_{k=p+1}^{\infty} k(k-1)(\omega(k+1)-\gamma)a_k|z|^{k-1}
\end{align*}
\]

Thus, \( f \in A(p, \gamma, \omega) \) and the theorem is established.

Note that, the sharpness follows if we choose the function \( f(z) \) as

\[
f(z) = z^p + \frac{p(p-1)(\omega(p+1)-\gamma)}{k(k-1)(k-p-\omega(k+1)+\gamma)}z^k,
\]

where \( k = p+1, p+2, \ldots \).

**Corollary 2.1.** Let \( f \in A(p, \gamma, \omega) \). Then

\[
a_k \leq \frac{p(p-1)(\omega(p+1)-\gamma)}{k(k-1)(k-p-\omega(k+1)+\gamma)},
\]

where \( k = p+1, p+2, \ldots \)  \hspace{1cm} (7)

### 2.2. Growth and Distortion.

A lower and upper bound of \( |f(z)| \) and \( |f'(z)| \) will be considered by the following theorems respectively, where the bounds for the function \( f(z) \) of the form

\[
f(z) = z^p + \frac{(p-1)(\omega(p+1)-\gamma)}{(p+1)(1-\omega(p+2)+\gamma)}z^{p+1},
\]

**Theorem 2.2.1.** If the function \( f \in A(p, \gamma, \omega) \) that defined in (2), then

\[
\begin{align*}
&\frac{(p-1)(\omega(p+1)-\gamma)}{(p+1)(1-\omega(p+2)+\gamma)} \leq |f(z)| \\
&\leq \frac{(p-1)(\omega(p+1)-\gamma)}{(p+1)(1-\omega(p+2)+\gamma)}
\end{align*}
\]
for $0 < |z| = r, r < 1$.

Proof: Since $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, then

$$|f(z)| = \left| z^p + \sum_{k=p+1}^{\infty} a_k z^k \right| \leq |z|^p + |z|^{p+1} \sum_{k=p+1}^{\infty} a_k.$$  

From Theorem (2.1.1), we get:

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{(p-1)[\omega(p+1) - \gamma]}{(p+1)[1 - \omega(p+2) + \gamma]}.$$  

Then

$$|f(z)| \leq r^p + r^{p+1} \frac{(p-1)[\omega(p+1) - \gamma]}{(p+1)[1 - \omega(p+2) + \gamma]},$$

and

$$|f(z)| \geq |z|^p - |z|^{p+1} \sum_{k=p+1}^{\infty} a_k = r^p - r^{p+1} \sum_{k=p+1}^{\infty} a_k.$$  

Hence

$$|f(z)| \geq r^p - r^{p+1} \frac{(p-1)[\omega(p+1) - \gamma]}{(p+1)[1 - \omega(p+2) + \gamma]}.$$  

So the proof is complete.  

Theorem 2.2.2. If the function $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ that defined in (2), then

$$pr^{p-1} - r^p \frac{(p-1)[\omega(p+1) - \gamma]}{[1 - \omega(p+2) + \gamma]} \leq |f'(z)| \leq pr^{p-1} + r^p \frac{(p-1)[\omega(p+1) - \gamma]}{[1 - \omega(p+2) + \gamma]},$$

for $0 < |z| = r, r < 1$.

Proof: since $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, then

$$|f'(z)| = \left| pz^{p-1} + \sum_{k=p+1}^{\infty} k a_k z^{k-1} \right| \leq p|z|^{p-1} + |z|^p \sum_{k=p+1}^{\infty} k a_k.$$  

From Theorem (2.1.1), we have

$$\sum_{k=p+1}^{\infty} k a_k \leq \frac{(p-1)[\omega(p+1) - \gamma]}{[1 - \omega(p+2) + \gamma]}.$$  

Thus

$$|f'(z)| \leq pr^{p-1} + r^p \frac{(p-1)[\omega(p+1) - \gamma]}{[1 - \omega(p+2) + \gamma]},$$

and

$$|f'(z)| \geq p|z|^{p-1} - |z|^p \sum_{k=p+1}^{\infty} k a_k.$$  

$$|f'(z)| \geq pr^{p-1} - r^p \frac{(p-1)[\omega(p+1) - \gamma]}{[1 - \omega(p+2) + \gamma]}.$$  

2.3 Radii of Starlikeness, Convexity and Close-to-Convexity.

The following theorems explain the radii of starlikeness, convexity and close-to-convexity.

Theorem 2.3.1. If the function $f(z) \in A(p, \gamma, \omega)$ that defined in (2). Then it is multivalent starlike of order $\delta$ ($0 \leq \delta < p$) in the disc $|z| < r_1$, where

$$r_1(p, \gamma, \omega, \delta) = \inf_k \left[ \frac{k(p-\delta)(k-1)[k-p-\omega(k+1) + \gamma]}{p(k-\delta)(p-1)[\omega(p+1) - \gamma]} \right]^{\frac{1}{p-\delta}},$$

$k \geq p+1$.

The result is sharp for the external function $f(z)$ given by (6).

Proof: It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta, \quad (0 \leq \delta < p),$$

for $|z| < r_1(p, \gamma, \omega, \delta)$.

We have

$$\left| \frac{zf'(z)}{f(z)} - p \right| = \left| \frac{z[pz^{p-1} + \sum_{k=p+1}^{\infty} k a_k z^{k-1}] - p[z^p + \sum_{k=p+1}^{\infty} a_k z^k]}{z^p + \sum_{k=p+1}^{\infty} a_k z^k} \right| \leq \frac{\left| \sum_{k=p+1}^{\infty} \left( k - p \right) a_k |z|^{k-p} \right|}{\left| 1 - \sum_{k=p+1}^{\infty} a_k |z|^{k-p} \right|}.$$
Thus
\[ \left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta. \]

If
\[ \sum_{k=p+1}^{\infty} \frac{(k-\delta)a_k|z|^{k-p}}{(p-\delta)} \leq 1. \] (8)

Therefore by Corollary (2.1.1), inequality (8) is true if:
\[ \frac{(k-\delta)|z|^{k-p}}{(p-\delta)} \leq \frac{k(k-1)(k-p-\omega(k+1)+\gamma)}{p(p-1)[\omega(p+1)-\gamma]}, \]
equivalently if:
\[ |z| \leq \left[ \frac{k(k-1)(k-p-\omega(k+1)+\gamma)}{p(p-1)[\omega(p+1)-\gamma]} \right]^{\frac{1}{k-p}}. \] (9)

The theorem follows from (9) ■

Theorem 2.3.2. If the function \( f(z) \) defined in (2) be in the class \( A(p, \gamma, \omega) \) where
\[ r_2(p, \gamma, \omega, \delta) = \inf_k \left[ \frac{(p-\delta)(k-1)[k-p-\omega(k+1)+\gamma]}{p(p-1)[\omega(p+1)-\gamma]} \right]^{\frac{1}{k-p}}, (k \geq p + 1). \]

The result is sharp for the external function \( f(z) \) given by (6).

Proof: It is sufficient to show that
\[ \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p - \delta, \quad (0 \leq \delta < p), \quad \text{for} \quad |z| < r_2(p, \gamma, \omega, \delta). \]

We have
\[ \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq \sum_{k=p+1}^{\infty} k(k-\delta)a_k|z|^{k-p}. \]

Thus
\[ \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p - \delta, \]
equivalently if
\[ \sum_{k=p+1}^{\infty} \frac{k(k-\delta)a_k|z|^{k-p}}{(p-\delta)} \leq 1. \]

Therefore by Corollary (2.1.1), last inequality is true if:
\[ \frac{k(k-\delta)|z|^{k-p}}{(p-\delta)} \leq \frac{k(k-1)(k-p-\omega(k+1)+\gamma)}{p(p-1)[\omega(p+1)-\gamma]}, \]
equivalently if
\[ |z| \leq \left[ \frac{(k-1)(p-\delta)[k-p-\omega(k+1)+\gamma]}{p(p-1)[\omega(p+1)-\gamma]} \right]^{\frac{1}{k-p}}. \] (10)

The theorem follows from (10) ■

Theorem 2.3.3. Let the function \( f(z) \) defined by (2) be in the class \( A(p, \gamma, \omega) \). Then \( f(z) \) is multivalent close-to-convex of order \( \delta \) \((0 \leq \delta < p)\) in the disc \(|z| < r_2\), where
\[ r_2(p, \gamma, \omega, \delta) = \inf_k \left[ \frac{(k-1)(p-\delta)[k-p-\omega(k+1)+\gamma]}{p(p-1)[\omega(p+1)-\gamma]} \right]^{\frac{1}{k-p}}, (k \geq p + 1). \]

The result is sharp for the external function \( f(z) \) given by (6).

Proof: We must show that:
\[ \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \delta, \quad (0 \leq \delta < p), \]
for \(|z| < r_3(p, \gamma, \omega, \delta)\).

We have:
\[ |f(z)|_{z^{p-1}} - p \leq \sum_{k=p+1}^{\infty} k_{a_k}|z|^{k-p}. \]

Thus
\[ \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \delta, \]
equivalently if
\[ \sum_{k=p+1}^{\infty} \frac{k_{a_k}|z|^{k-p}}{(p-\delta)} \leq 1. \]

Hence by Corollary (2.1.1), the last statement will be true if:
\[ \frac{k|z|^{k-p}}{(p-\delta)} \leq \frac{k(k-1)(k-p-\omega(k+1)+\gamma)}{p(p-1)[\omega(p+1)-\gamma]}, \]
equivalently if
\[ |z| \leq \left[ \frac{(k-1)(p-\delta)[k-p-\omega(k+1)+\gamma]}{p(p-1)[\omega(p+1)-\gamma]} \right]^{\frac{1}{k-p}}. \] (11)

The theorem follows easily from (11) ■

2.4. Extreme Points.

The following theorem discuss the extreme points of the class \( A(p, \gamma, \omega) \).
Theorem 2.4.1. Let \( f_p(z) = z^p \) and \( f_k(z) = z^p + \frac{p(p-1)[\omega(p+1)-\gamma]}{k(k-1)[k-p-\omega(k+1)+\gamma]}z^k \),

where \( k \geq p+1, p \geq 1, \frac{1}{2} \leq \omega < 1, 0 < \gamma \leq \frac{1}{2} \).

Then the function \( f \) belongs to the class \( A(p, \gamma, \omega) \) if and only if it can be written as:

\[
f(z) = \mathcal{L}_p z^p + \sum_{k=p+1}^{\infty} \mathcal{L}_k f_k(z),
\]

such that

\[
(\mathcal{L}_p \geq 0, \mathcal{L}_k \geq 0, k \geq p+1) \quad \text{and} \quad \mathcal{L}_p + \sum_{k=p+1}^{\infty} \mathcal{L}_k = 1
\]

Proof: Suppose that \( f(z) \) that defined in (12). Then

\[
f(z) = \mathcal{L}_p z^p + \sum_{k=p+1}^{\infty} \mathcal{L}_k \left[ z^p + \frac{p(p-1)[\omega(p+1)-\gamma]}{k(k-1)[k-p-\omega(k+1)+\gamma]}z^k \right]
\]

\[
= z^p + \sum_{k=p+1}^{\infty} \frac{p(p-1)[\omega(p+1)-\gamma]}{k(k-1)[k-p-\omega(k+1)+\gamma]}\mathcal{L}_k z^k.
\]

Hence

\[
\sum_{k=p+1}^{\infty} \frac{k(k-1)[k-p-\omega(k+1)+\gamma]}{p(p-1)[\omega(p+1)-\gamma]} \times \frac{p(p-1)[\omega(p+1)-\gamma]}{k(k-1)[k-p-\omega(k+1)+\gamma]} \mathcal{L}_k = 1 - \mathcal{L}_p \leq 1.
\]

Thus \( f \in A(p, \gamma, \omega) \).

Conversely, suppose that \( f \in A(p, \gamma, \omega) \), we may set

\[
\mathcal{L}_k = \frac{k(k-1)[k-p-\omega(k+1)+\gamma]}{p(p-1)[\omega(p+1)-\gamma]} a_k,
\]

where \( a_k \) is defined in (5). Then

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k
\]

This complete the proof of Theorem (7).

3. Convolution Properties

The following theorems shows the convolution properties for the functions in the class \( A(p, \gamma, \omega) \).

Theorem 3.1 Let the functions \( f_r(z) \in A(p, \gamma, \omega) \) such that

\[
f_r(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,r} z^k, \quad (a_{k,r} \geq 0, \quad r = 1, 2).
\]

Then \( (f_1 * f_2) \in A(p, \gamma, d) \), where

\[
d \geq \frac{p(p-1)[\omega p + 1 - \gamma]^2 (k-p+\gamma) + \gamma k(k-1)[k-p-(\omega k+1)+\gamma]^2}{k(p+1)[k-1][k-p-(\omega k+1)+\gamma]^2 + p(k+1)[p-1][\omega p + 1 - \gamma]^2}.
\]

The result is sharp for the functions \( f_r \) \( (r = 1, 2) \) given by (6).

Proof: We will find the smallest \( d \) such that

\[
\sum_{k=p+1}^{\infty} \frac{k(k-1)[k-p-d(k+1)+\gamma]}{p(p-1)[d(p+1)-\gamma]} a_{k,1} a_{k,2} \leq 1
\]

Since \( f_r \in A(p, \gamma, \omega) \), \( (r = 1, 2) \), then

\[
\sum_{k=p+1}^{\infty} \frac{k(k-1)[k-p-(\omega k+1)+\gamma]}{p(p-1)[(\omega p + 1) - \gamma]} a_{k,r} \leq 1, \quad (r = 1, 2)
\]

By Cauchy-Schwarz inequality, we get

\[
\sum_{k=p+1}^{\infty} \frac{k(k-1)[k-p-(\omega k+1)+\gamma]}{p(p-1)[(\omega p + 1) - \gamma]} \sqrt{a_{k,1} a_{k,2}} \leq 1. \quad (14)
\]
Now, we need only to show that:
\[
\frac{k(k - 1)[k - p - d(k + 1) + \gamma]}{p(p - 1)[d(p + 1) - \gamma]} a_{k,1} a_{k,2}
\]
\[
\leq \frac{k(k - 1)[k - p - (ak + 1) + \gamma]}{p(p - 1)[(ap + 1) - \gamma]} \sqrt{a_{k,1} a_{k,2}},
\]
and this equivalently to:
\[
\sqrt{a_{k,1} a_{k,2}} \leq \frac{[d(p + 1) - \gamma][k - p - (ak + 1) + \gamma]}{[k - p - d(k + 1) + \gamma]((ap + 1) - \gamma)}.
\]

From (14), we have
\[
\sqrt{a_{k,1} a_{k,2}} \leq \frac{p(p - 1)((ap + 1) - \gamma)}{k(k - 1)[k - p - (ak + 1) + \gamma]].
\]
Thus, it is sufficient to show that
\[
\frac{p(p - 1)((ap + 1) - \gamma)}{k(k - 1)[k - p - (ak + 1) + \gamma]} \leq \frac{[d(p + 1) - \gamma][k - p - (ak + 1) + \gamma]}{[k - p - d(k + 1) + \gamma]((ap + 1) - \gamma)}.
\]

which implies to

Thus, the theorem is established \(\blacksquare\)

**Theorem 3.2.** Let the functions \(f_r(z)\) in Theorem 3.1 belongs to the class \(A(p, \gamma, \omega)\). Then the function \(h(z) = z^p + \sum_{k=p+1}^{\infty}(a_{k,1}^2 + a_{k,2}^2)z^k\), belongs also to the class \(A(p, \gamma, \omega)\)

where
\[
p(p + 1)[1 - (ap(p + 1) + 1) + \gamma] - 2p(p - 1)[(ap + 1) - \gamma] \geq 0.
\]

Proof: Since \(f_1(z) \in A(p, \gamma, \omega)\), we get
\[
\sum_{k=p+1}^{\infty} \left[ \frac{k(k - 1)[k - p - (ak + 1) + \gamma]}{p(p - 1)((ap + 1) - \gamma)} \right]^2 a_{k,1}^2
\]
\[
\leq \left( \sum_{k=p+1}^{\infty} \frac{k(k - 1)[k - p - (ak + 1) + \gamma]}{p(p - 1)[(ap + 1) - \gamma]} a_{k,1} \right)^2,
\]
\[
\leq 1, \quad (15)
\]

and
\[
\sum_{k=p+1}^{\infty} \left[ \frac{k(k - 1)[k - p - (ak + 1) + \gamma]}{p(p - 1)[(ap + 1) - \gamma]} \right]^2 a_{k,1}^2
\]
\[
\leq \left( \sum_{k=p+1}^{\infty} \frac{k(k - 1)[k - p - (ak + 1) + \gamma]}{p(p - 1)[(ap + 1) - \gamma]} a_{k,1} \right)^2,
\]
\[
\leq 1. \quad (16)
\]
Combining the inequalities (15) and (16), gives
\[
\sum_{k=p+1}^{\infty} \frac{1}{2} \left[ \frac{k(k - 1)[k - p - (ak + 1) + \gamma]}{p(p - 1)((ap + 1) - \gamma)} \right]^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1. \quad (17)
\]

According to Theorem (2.1), it is sufficient to show that:
\[
\sum_{k=p+1}^{\infty} \left[ \frac{k(k - 1)[k - p - (ak + 1) + \gamma]}{p(p - 1)((ap + 1) - \gamma)} \right]^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1.
\]

Thus the last inequality, will be satisfies if, for \(k = p + 1, p + 2, p + 3, ...\)
\[
\left[ \frac{k(k - 1)[k - p - (ak + 1) + \gamma]}{p(p - 1)((ap + 1) - \gamma)} \right] \leq \frac{1}{2} \left[ \frac{k(k - 1)[k - p - (ak + 1) + \gamma]}{p(p - 1)((ap + 1) - \gamma)} \right]^2.
\]

Or if
\[
k(k - 1)[k - p - (ak + 1) + \gamma] - 2p(p - 1)((ap + 1) - \gamma) \geq 0. \quad (18)
\]

For \((k = p + 1, p + 2, p + 3, ...)\) the left hand side of (18) is increasing function of \(k\), hence it is satisfied for all \(k\) if:
\[
p(p + 1)[1 - (ap(p + 1) + 1) + \gamma] - 2p(p - 1)((ap + 1) - \gamma) \geq 0.
\]

which is true by our assumption. Therefor the prove is complete \(\blacksquare\)

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REFERENCES


